Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Geometry and Physics 53 (2005) 345–353

JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

# A class of Osserman spaces

Dmitri V. Alekseevsky[a], Novica Blažić[b], Vicente Cortés[c,*], Srdjan Vukmirović[b]

[a] *Department of Mathematics, University of Hull, Cottingham Road, Hull HU6 7RX, UK*
[b] *Departement of Mathematics, University of Belgrade, Studentski trg 16, p.p. 550, 11000 Belgrade, Yugoslavia*
[c] *Institut de Mathématiques Élie Cartan, Université Henri Poincaré—Nancy I, Faculté des Sciences, B.P. 239, 54506 Vandoeuvre-lès-Nancy, France*

Received 17 April 2004; received in revised form 30 June 2004; accepted 16 July 2004
Available online 11 September 2004

**Abstract**

We prove that a symmetric space is Osserman if its complexification is a complex hyper-Kähler symmetric space. This includes all pseudo-hyper-Kähler as well as para-hyper-Kähler symmetric spaces. We extend the classification of pseudo-hyper-Kähler symmetric spaces obtained by the first and the third author to the class of para-hyper-Kähler symmetric spaces. These manifolds are possible targets for the scalars of rigid $N = 2$ supersymmetric field theories with hypermultiplets on four-dimensional space-times with Euclidean signature.


*MSC 2000:* C35; 53C26; 53C25; 53C50

*Keywords:* Symmetric spaces; Hyper-Kähler manifolds; Para-hyper-Kähler manifolds; Osserman spaces

## 1. Introduction

A (pseudo-) Riemannian manifold is called an *Osserman space* if the characteristic polynomial of the Jacobi operator $R_X = R(\cdot, X)X$ is same for all unit vectors $X$. In this paper

* Corresponding author.
*E-mail addresses:* d.v.alekseevsky@maths.hull.ac.uk (D.V. Alekseevsky), cortes@iecn.u-nancy.fr (V. Cortés), blazicn@matf.bg.ac.yu (N. Blažić), vsrdjan@matf.bg.ac.yu (S. Vukmirovi).

0393-0440/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2004.07.004

we present a new class of Ossermann spaces with many interesting properties (see Theorem 4). Our examples are pseudo-Riemannian symmetric spaces $M = G/H$, where $G$ is a 3-step nilpotent group and $H = \mathrm{Hol}(M)$ is Abelian. Moreover, they carry an invariant hyper-Kähler or para-hyper-Kähler structure. Simply connected symmetric hyper-Kähler manifolds were classified in [2]. Here we give also the classification of simply connected symmetric para-hyper-Kähler manifolds (see Theorem 3). We believe that this classification will be useful for physical applications, since there is evidence that para-hyper-Kähler manifolds are precisely the allowed targets for the scalars of rigid $N = 2$ supersymmetric field theories with hypermultiplets on four-dimensional space-times with Euclidean signature (see [4]).

## 2. Basic facts about symmetric spaces

In this section we recall some general facts about symmetric spaces.

### 2.1. Real and complex symmetric spaces

A *pseudo-Riemannian symmetric space* is a pseudo-Riemannian manifold $(M, g)$ such that any point is an isolated fixed point of an isometric involution. Such a pseudo-Riemannian manifold admits a transitive Lie group of isometries $L$ and can be identified with $L/L_o$, where $L_o$ is the stabilizer of a point $o$. More precisely, any simply connected pseudo-Riemannian symmetric space $M = G/K$ is associated with a symmetric decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k} \tag{2.1}$$

of the Lie algebra $\mathfrak{g} = \mathit{Lie}\, G$ together with an $\mathrm{Ad}_K$-invariant pseudo-Euclidean scalar product on $\mathfrak{m}$. We will assume that $G$ acts almost effectively on $M$, i.e. $\mathfrak{k}$ does not contain any nontrivial ideal of $\mathfrak{g}$, that $M$ and $G$ are simply connected and that $K$ is connected. Then, under the natural identification of the tangent space $T_oM$ at the canonical base point $o = eK$ with $\mathfrak{m}$, the holonomy group $Hol \subset \mathrm{Ad}_K|\mathfrak{m}$. We will denote by $\mathfrak{h}$ the holonomy Lie algebra, which is spanned by the curvature operators $R(x, y)$, $x, y \in \mathfrak{m}$. Recall that the curvature tensor $R$ of a symmetric space $M = G/K$ at $o$ is $\mathrm{ad}_{\mathfrak{k}}$-invariant and is given by

$$R(x, y) = -\mathrm{ad}_{[x,y]}|_{\mathfrak{m}}.$$

Since the isotropy representation is faithful, we can identify the holonomy algebra $\mathfrak{h} = \mathrm{span}\{R(x, y)|x, y \in \mathfrak{m}\}$ with the subalgebra $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$ of $\mathfrak{g}$. It is easy to see that the group generated by the ideal $\mathfrak{h} + \mathfrak{m} \subset \mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ acts transitively on $M$. So replacing the symmetric decomposition $\mathfrak{k} + \mathfrak{m}$ by $\mathfrak{h} + \mathfrak{m}$, if necessary, we can assume that $\mathfrak{k} = \mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$.

Note that the symmetric decomposition can be reconstructed from the curvature tensor $R$. More precisely, let $\mathfrak{k} \subset \mathrm{gl}(\mathfrak{m})$ be a linear Lie algebra which preserves a pseudo-Euclidean scalar product on $\mathfrak{m}$ and let $R$ be a $\mathfrak{k}$-invariant $\mathfrak{k}$-valued 2-form on $\mathfrak{m}$ which satisfies the Bianchi identity. Then the formulas

$$[A, x] = Ax \quad \text{for} \quad A \in \mathfrak{k}, x \in \mathfrak{m} \quad \text{and} \quad [x, y] = -R(x, y) \quad \text{for} \quad x, y \in \mathfrak{m}$$

define the structure of a Lie algebra with a symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. The corresponding simply connected pseudo-Riemannian symmetric space $M = G/K$ has the holonomy algebra $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$.

A *complex Riemannian manifold* is a complex manifold $M$ equipped with a complex metric $g$, i.e. a holomorphic section $g \in \Gamma(S^2T^*M)$ which defines a nondegenerate complex quadratic form. As in the real case any such manifold has a unique holomorphic torsionfree and metric connection (Levi-Civita connection).

A *complex Riemannian symmetric space* is a complex Riemannian manifold $(M, g)$ such that any point is an isolated fixed point of an isometric holomorphic involution. Like in the real case one can prove that it admits a transitive complex Lie group of holomorphic isometries and that any simply connected complex Riemannian symmetric $M$ is associated to a complex symmetric decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] = \mathfrak{k} \tag{2.2}$$

of a complex Lie algebra $\mathfrak{g}$ together with an $\mathrm{ad}_{\mathfrak{k}}$-invariant complex scalar product on $\mathfrak{m}$. More precisely $M = G/K$, where $G$ is the simply connected complex Lie group with the Lie algebra $\mathfrak{g}$ and $K$ is the (closed) connected subgroup associated with $\mathfrak{k}$. The holonomy group of such manifold is $H = \mathrm{Ad}_K|\mathfrak{m}$.

Any pseudo-Riemannian symmetric space $M = G/K$ associated with a symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ has a canonical complexification $M^{\mathbb{C}} = G^{\mathbb{C}}/K^{\mathbb{C}}$ defined by the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{\mathbb{C}}$ of the symmetric decomposition.

### 2.2. *Hyper-Kähler and para-hyper-Kähler symmetric spaces*

A (possibly indefinite) $(\varepsilon)$-*hyper-Kähler manifold* is a pseudo-Riemannian manifold $(M^{4n}, g)$ together with a parallel $(\varepsilon)$-*hypercomplex structure*, i.e. three anticommuting parallel endomorphism fields $(J_1, J_2, J_3 = J_1J_2)$, which are skew symmetric with respect to $g$ and which satisfy $J_1^2 = J_2^2 = \varepsilon\mathrm{Id}$, $\varepsilon \in \{+, -\}$. Notice that the product $J_3$ is a parallel skew symmetric complex structure. The notion of $(\varepsilon)$-hyper-Kähler manifold unifies the notion of *hyper-Kähler manifold* $(\varepsilon = -)$ and of *para-hyper-Kähler manifold* $(\varepsilon = +)$. Notice that the triple $(J_1, J_2, J_3)$ defines the structure of a vector space over the quaternions $(\varepsilon = -)$ or over the para-quaternions $(\varepsilon = +)$ on the tangent space.

The above conditions mean that the holonomy group $\mathrm{Hol} \subset \mathrm{Sp}(k, l)$, $n = k + l$, in the case $\varepsilon = -$ and $\mathrm{Hol} \subset \mathrm{Sp}(n, \mathbb{R}) := \mathrm{Sp}(\mathbb{R}^{2n})$ in the case $\varepsilon = +$. Two $(\varepsilon)$-hyper-Kähler manifolds $(M, g, J_\alpha)$ $(\alpha = 1, 2, 3)$ and $(M', g', J'_\alpha)$ are called *isomorphic* if there is a diffeomorphism $\varphi : M \to M'$ such that $\varphi^* J'_\alpha = J_\alpha$ and $\varphi^* g' = g$.

An $(\varepsilon)$-*hyper-Kähler symmetric space* is a pseudo-Riemannian symmetric space $(M = G/K, g)$ together with an invariant $(\varepsilon)$-hypercomplex structure. Consider now a simply connected $(\varepsilon)$-hyper-Kähler symmetric space $(M = G/K, g, J_\alpha)$. Without restriction of generality we will assume that $G$ acts almost effectively. $M$ being $(\varepsilon)$-hyper-Kähler is equivalent to

$$\mathrm{Ad}_K|\mathfrak{m} \subset \begin{cases} \mathrm{Sp}(k, l), & \text{for} \quad \varepsilon = - \\ \mathrm{Sp}(n, \mathbb{R}), & \text{for} \quad \varepsilon = +. \end{cases}$$

Since $K$ is connected, this condition means precisely that $\mathrm{ad}_{\mathfrak{k}}|\mathfrak{m} \subset \mathrm{so}(\mathfrak{m})$ commutes with the Lie algebra

$$Q := \mathrm{span}\{J_1, J_2, J_3\} = \begin{cases} \mathrm{sp}(1) \subset \mathrm{so}(\mathfrak{m}) = \mathrm{so}(4k, 4l), & \text{for} \quad \varepsilon = - \\ \mathrm{sl}(2, \mathbb{R}) \subset \mathrm{so}(\mathfrak{m}) = \mathrm{so}(2n, 2n), & \text{for} \quad \varepsilon = + \end{cases}$$

spanned by the three anticommuting structures $J_1, J_2, J_3$.

A *complex hyper-Kähler manifold* is a complex Riemannian manifold $(M^{4n}, g)$ of complex dimension $4n$ together with a compatible hypercomplex structure, i.e. three $g$-orthogonal parallel complex linear endomorphisms $(J_1, J_2, J_3 = J_1 J_2)$ with $J_\alpha^2 = -1$. This means that the holonomy group $\mathrm{Hol} \subset \mathrm{Sp}(n, \mathbb{C}) = Z_{O(4n,\mathbb{C})}(\mathrm{Sp}(1, \mathbb{C}))$. The linear group $\mathrm{Sp}(n, \mathbb{C})$ is diagonally embedded into $\mathrm{Sp}(n, \mathbb{C}) \times \mathrm{Sp}(n, \mathbb{C}) \subset \mathrm{GL}(4n, \mathbb{C})$. Two complex hyper-Kähler manifolds $(M, g, J_\alpha)$ $(\alpha = 1, 2, 3)$ and $(M', g', J'_\alpha)$ are called *isomorphic* if there exists a holomorphic diffeomorphism $\varphi : M \to M'$ such that $\varphi^* J'_\alpha = J_\alpha$ and $\varphi^* g' = g$. We notice that the complexification of an $(\varepsilon)$-hyper-Kähler symmetric space is a complex hyper-Kähler symmetric space.

## 3. Classification of real and complex hyper-Kähler symmetric spaces

Now we recall the classification of real and complex hyper-Kähler symmetric spaces [AC]. Let $(E, \omega)$ be a complex symplectic vector space of dimension $2n$ and $E = E^+ \oplus E^-$ a Lagrangian decomposition. Then any element $S \in S^4 E^+$ defines a simply connected complex symmetric space $M_S^c$ of dimension $4n$ which is associated with the symmetric decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{m} = H \otimes E$, $H = \mathbb{C}^2 = \mathbb{C}1 \oplus \mathbb{C}j = \mathbb{H} = \mathrm{span}\{1, i, j, k\}$ with its standard complex symplectic structure $\omega^H$, $\mathfrak{h} = \mathrm{span}\{S_{e,e'} |\, e, e' \in E\} \subset \mathrm{sp}(E)$ with the natural action on $\mathfrak{m} \subset H \otimes E$ and the Lie bracket $\mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{h}$ is given by

$$[h \otimes e, h' \otimes e'] = \omega^H(h, h') S_{e,e'}.$$

Here $S_{e,e'} \in S^2 E = \mathrm{sp}(E)$ denotes the contraction of $S \in S^4 E^+ \subset S^4 E$ with $ee' \in S^2 E$ by means of $\omega$.

**Theorem 1.** *[AC] Let $M_S^c$ be the complex symmetric space associated to $S \in S^4 E^+$. Then it is a complex hyper-Kähler symmetric space with complex Riemannian metric $g$ defined by $\omega^H \otimes \omega$. and compatible hypercomplex structure $(J_1, J_2, J_3)$ defined by $(R_i \otimes \mathrm{Id}, R_j \otimes \mathrm{Id}, -R_k \otimes \mathrm{Id})$, where $R_q$ denotes the right multiplication by the quaternion $q$. Moreover, $M_S^c$ has no flat factor if and only if $S_{E,E} E := \mathrm{span}\{S_{e,e'} e'' | e, e', e'' \in E\} = E^+$. Conversely, any simply connected complex hyper-Kähler symmetric space is of the form $M_S^c$.*

To obtain real hyper-Kähler symmetric spaces, let us assume that on $(E, \omega)$ we have a compatible quaternionic structure $j : E \to E$, i.e. an antilinear map such that $j^2 = -\mathrm{Id}$ and $j^*\omega = \bar{\omega}$, and a $j$-invariant Lagrangian decomposition $E = E^+ \oplus E^-$. Such a decomposition exists if and only if the Hermitian form $\gamma = \omega(\cdot, j\cdot)$ has real signature $(4m, 4m)$, where $m$ is related to the complex dimension $4n$ of $M_S^c$ by $n = 2m$. On $H = \mathbb{H}$ we fix the standard quaternionic structure $j^H = L_j$, the left-multiplication with $j$. It satisfies $(j^H)^*\omega^H = \bar{\omega}^H$. We denote by $\tau$ the real structure on $S^{2r}E$ induced by the quaternionic structure $j$ on $E$: $\tau(e_1 e_2 \ldots e_{2r}) := j(e_1)j(e_2)\ldots j(e_{2r})$, $e_i \in E$. On $\mathfrak{m} = H \otimes E$ we have the real structure $\rho := j^H \otimes j$. We assume that $S \in S^4E^+$ is real, i.e. $\tau S = S$. Then $S$ defines a (real) symmetric space $M_S$, which is associated with the symmetric decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 + \mathfrak{m}_0,$$

where

$$\mathfrak{h}_0 := \mathfrak{h}^\tau := \mathfrak{h} \cap (S^2E)^\tau, \quad \mathfrak{m}_0 := \mathfrak{m}^\rho = (H \otimes E)^\rho.$$

**Theorem 2.** *[AC] Let $M_S$ be the symmetric space associated to $S \in (S^4E^+)^\tau$. Then it is a (real) hyper-Kähler symmetric space with pseudo-Riemannian metric defined by $(\omega^H \otimes \omega)|_{\mathfrak{m}_0}$ and compatible hypercomplex structure $(J_1, J_2, J_3)$ defined by $J_1 := (R_i \otimes \mathrm{Id})|_{\mathfrak{m}_0}$, $J_2 := (R_j \otimes \mathrm{Id})|_{\mathfrak{m}_0}$ and $J_3 := J_1J_2 = -(R_k \otimes \mathrm{Id})|_{\mathfrak{m}_0}$. The metric has signature $(4m, 4m)$, $4m = \dim_{\mathbb{C}} E$. The complexification of $M_S$ coincides with the complex hyper-Kähler symmetric space $M_S^c$ associated to $S \in S^4E^+$. Any simply connected hyper-Kähler symmetric space is the Riemannian product of a flat hyper-Kähler symmetric space (a hyper-Hermitian vector space) of arbitrary signature $(4p, 4q)$ and a hyper-Kähler symmetric space of the form $M_S$.*

## 4. Classification of para-hyper-Kähler symmetric spaces

In this section we give the classification of simply connected para-hyper-Kähler symmetric spaces. The basic data for the construction of such spaces are the following: $(E_0, \omega_0)$ a real symplectic vector space of real dimension $2n$, $E_0 = E_0^+ \oplus E_0^-$ a Lagrangian decomposition and $H_0 = \mathbb{R}^2$ with its standard symplectic structure $\omega_0^H$ and para-hypercomplex structure

$$j_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad j_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad j_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then any element $S \in S^4E_0^+$ defines a simply connected symmetric space $M_S$ of dimension $4n$ which is associated with the symmetric decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 + \mathfrak{m}_0,$$

where $\mathfrak{m} = H_0 \otimes E_0$, $\mathfrak{h} = \text{span}\{S_{e,e'}|e, e' \in E_0\} \subset \text{sp}(E_0)$ with the natural action on $\mathfrak{m}_0 = H_0 \otimes E_0$ and the Lie bracket $\mathfrak{m}_0 \wedge \mathfrak{m}_0 \to \mathfrak{h}_0$ is given by

$$[h \otimes e, h' \otimes e'] = \omega_0^H(h, h')S_{e,e'}.$$

Here $S_{e,e'} \in S^2E_0 = \text{sp}(E_0)$ denotes the contraction of $S \in S^4E_0^+ \subset S^4E_0$ with $ee' \in S^2E_0$ by means of $\omega_0$.

**Proposition 1.** *The Lie algebra $\mathfrak{g}_0$ associated with $S \in S^4E_0^+$ is 3-step nilpotent if $S \neq 0$ and Abelian if $S = 0$.*

**Proof.** To $S \in S^4E_0^+$ we associate the subspace

$$\Sigma_S := S_{E_0,E_0}E_0 = \text{span}\{S_{e,e'}e''|e, e', e'' \in E_0\} \subset E_0^+,$$

which is nontrivial for $S \neq 0$. Using the fact that $\mathfrak{h}_0$ is Abelian and that $S_{e,e'}e'' = 0$ if one of the three arguments belongs to $E_0^+$, one can easily check that the central series is given by:

$$[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{h}_0 + H_0 \otimes \Sigma_S$$

$$[\mathfrak{g}_0, [\mathfrak{g}_0, \mathfrak{g}_0]] = H_0 \otimes \Sigma_S$$

$$[\mathfrak{g}_0, [\mathfrak{g}_0, [\mathfrak{g}_0, \mathfrak{g}_0]]] = 0.$$

□

**Theorem 3.** *Let $M_S$ be the symmetric space associated to $S \in S^4E_0^+$. Then it is a para-hyper-Kähler symmetric space with pseudo-Riemannian metric defined by $\omega_0^H \otimes \omega_0$ and compatible para-hypercomplex structure $(J_1, J_2, J_3)$ defined by $J_1 := j_1 \otimes \text{Id}$, $J_2 := j_2 \otimes \text{Id}$ and $J_3 := J_1J_2 = j_3 \otimes \text{Id}$. The metric has signature $(2n, 2n)$, $2n = \dim E_0$. The complexification of $M_S$ coincides with the complex hyper-Kähler symmetric space $M^c_{S^c}$ associated to the complex extension $S^c \in S^4E^+$, $E^+ := E_0^+ \otimes \mathbb{C}$, of $S \in S^4E_0^+$. Any simply connected para-hyper-Kähler symmetric space is of the form $M_S$.*

**Proof.** It is easy to check that $M_S$ is a para-hyper-Kähler symmetric space of signature $(2n, 2n)$ whose complexification coincides with the complex hyper-Kähler symmetric space $M^c_{S^c}$. We now prove that any simply connected para-hyper-Kähler symmetric space is of the form $M_S$. Let $M$ be such a symmetric space associated with a symmetric decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 + \mathfrak{m}_0.$$

Since $M$ is para-hyper-Kähler , we can identify $\mathfrak{m}_0 = H_0 \otimes E_0$ with the tensor product of two symplectic vector spaces $H_0 = \mathbb{R}^2$ and $E_0$. The pseudo-Riemannian metric on $M$ corresponds to the product of the two symplectic structures and the holonomy Lie algebra

$\mathfrak{h}_0 \subset \mathrm{Id} \otimes \mathrm{sp}(E_0) = \mathrm{sp}(E_0)$ acts only on the second factor. The complexification $M^c$ of $M$ is the complex hyper-Kähler symmetric space associated to the symmetric decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad \mathfrak{h} := \mathfrak{h}_0 \otimes \mathbb{C}, \quad \mathfrak{m} := \mathfrak{m}_0 \otimes \mathbb{C} = H \otimes E,$$

where $(E, \omega)$ and $(H, \omega^H)$ are the complexifications of the real symplectic vector spaces $(E_0, \omega_0)$ and $(H_0, \omega_0^H)$, respectively. By Theorem 3, $M^c = M_{S^c}$ is associated to a quartic polynomial $S^c \in S^4E^+ \subset S^4E = S^4E^*$, where $E^+ \subset E$ is a Lagrangian subspace. Now we prove that the multilinear form $S^c \in S^4E^*$ has real values on $S^4E_0$ and hence defines an element of $S^4E_0$. First we remark that by the definition of the Lie bracket $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{h}$

$$\begin{aligned} S^2E_0 &= \mathrm{sp}(E_0) \supset \mathfrak{h}_0 = [\mathfrak{m}_0, \mathfrak{m}_0] = \mathrm{span}\{S^c_{e,e'}\omega^H(h,h') | e, e' \in E_0,\ h, h' \in H_0\} \\ &= \mathrm{span}\{S^c_{e,e'} | e, e' \in E_0\} =: S^c_{E_0,E_0}. \end{aligned}$$

This shows that $S^c_{E_0,E_0} \subset S^2E_0$ and hence that $S^c$ is real valued. We denote by $S$ the corresponding element of $S^4E_0$. Then we have $[h \otimes e, h' \otimes e'] = S_{e,e'}\omega_0^H(h,h')$ for all $e, e' \in E_0$ and $h, h' \in H_0$ and $\mathfrak{h}_0 = [\mathfrak{m}_0, \mathfrak{m}_0] = S_{E_0,E_0}$. The (real) subspace $E_0 \cap E^+ \subset E_0$ is isotropic. Therefore, there exists a Lagrangian subspace $E_0^+ \subset E_0$, which contains $E_0 \cap E^+$. Thus,

$$S_{E_0,E_0}E_0 = S^c_{E_0,E_0}E_0 \subset E_0 \cap E^+ \subset E_0^+.$$

Now the next lemma implies that $S \in S^4E_0^+$ and we can conclude that $M = M_S$. □

**Lemma 1.** *Let $E$ be a (real or complex) symplectic vector space, $F \subset E$ a subspace, $S \in S^4E$ and $S_{E,E}E \subset F$. Then $S \in S^4F$.*

**Proof.** Let $C \subset E$ be a complement of $F$. We denote by $C^\wedge$ and $F^\wedge$ the annihilators of $C$ and $F$ in $E^*$, respectively. We can consider them again as subspaces of $E$ via the identification $E^* \cong E$ given by the symplectic form. The decomposition

$$S^4E = \oplus_{p+q=4} S^pC \cdot S^qF.$$

gives rise to a decomposition $S = \sum_{p+q=4} S^{p,q}$, where $S^{p,q} \in S^pC \cdot S^qF$. Now from

$$F \supset S_{E,E}E \supset S_{F^\wedge,F^\wedge}F^\wedge = S^{4,0}_{F^\wedge,F^\wedge}F^\wedge \oplus S^{3,1}_{F^\wedge,F^\wedge}F^\wedge,$$

$$S^{4,0}_{F^\wedge,F^\wedge}F^\wedge \subset C \quad \text{and} \quad S^{3,1}_{F^\wedge,F^\wedge}F^\wedge \subset F,$$

we conclude that $S^{4,0} = 0$. Then, similarly, by considering successively $S_{F^\wedge,F^\wedge}C^\wedge \subset F$, $S_{F^\wedge,C^\wedge}C^\wedge \subset F$ and $S_{C^\wedge,C^\wedge}C^\wedge \subset F$ we can conclude that $S^{3,1} = 0$, $S^{2,2} = 0$ and $S^{1,3} = 0$, respectively. This shows that $S = S^{0,4} \in S^4F$. □

## 5. Hyper-Kähler symmetric spaces are Osserman

Let $(M, g)$ be a pseudo-Riemannian (or complex Riemannian) manifold, $R$ its curvature tensor and $X \in T_pM$ a tangent vector. The symmetric operator $R_X : T_pM \ni Y \mapsto R(Y, X)X \in T_pM$ is called the *Jacobi operator* of $X$. We denote by $S(TM) := \{X \in TM | g(X, X) = \pm 1\}$ the bundle of unit vectors.

**Definition 1.** $(M, g)$ is called an *Osserman space* if the characteristic polynomial $P_X(t) = \det(t \operatorname{Id} - R_X)$ of $R_X$ is independent of $X \in S(TM)$.

**Theorem 4.** *Let $(M, g)$ be an $(\varepsilon)$-hyper-Kähler symmetric space (or a complex hyper-Kähler symmetric space) of dimension $4n$. Then $(M, g)$ is an Osserman space. More precisely, the product of any two Jacobi operators is zero. In particular, all Jacobi operators are nilpotent and $P_X(t) = t^{4n}$.*

**Proof.** Since the complexification of any $(\varepsilon)$-hyper-Kähler symmetric space is a complex hyper-Kähler symmetric space, it is sufficient to prove the theorem for complex hyper-Kähler symmetric spaces. By Theorem 1 any complex hyper-Kähler symmetric space $M$ is of the form $M = M_S$ for some $S \in S^4E^+$. It is sufficient to check that $R_X R_Y = 0$ for all $X, Y \in T_oM$, where $o$ is the canonical base point of the symmetric space $M_S$. Any tangent vector $X \in T_oM = \mathfrak{m} = H \otimes E$ can be decomposed as $X = \sum_{i=1}^{2} h_i \otimes e_i$, where $h_i \in H$ and $e_i \in E$. So $R_X$ is given by:

$$R_X = \sum_{i,j} R(\cdot, h_i \otimes e_i)h_j \otimes e_j.$$

This shows that it is sufficient to check that the product of any two operators of the form $R(\cdot, h \otimes e)h' \otimes e'$ is zero. Let $h_1, \ldots, h_4 \in H$ and $e_1, \ldots, e_4 \in E$. Applying the operator $R(\cdot, h_1 \otimes e_1)h_2 \otimes e_2$ to $h \otimes e$ we have:

$$R(h \otimes e, h_1 \otimes e_1)h_2 \otimes e_2 = \omega^H(h, h_1)h_2 \otimes S_{e,e_1}e_2 = \omega^H(h, h_1)h_2 \otimes S_{e_1,e_2}e.$$

Next we apply $R(\cdot, h_3 \otimes e_3)h_4 \otimes e_4$ to the result, which yields:

$$\begin{aligned}
&\omega^H(h, h_1)R(h_2 \otimes S_{e_1,e_2}e, h_3 \otimes e_3)h_4 \otimes e_4 \\
&\quad = \omega^H(h, h_1)\omega^H(h_2, h_3)h_4 \otimes S_{(S_{e_1,e_2}e),e_3}e_4 \\
&\quad = \omega^H(h, h_1)\omega^H(h_2, h_3)h_4 \otimes S_{e_3,e_4}S_{e_1,e_2}e = 0.
\end{aligned}$$

□

Here we have used the complete symmetry of $S$, the fact that $S_{e_1,e_2}e \in E^+$ and that $S_{E,E} \subset S^2E^+$ vanishes on $E^+$. This shows that the composition of any two operators of the form $R(\cdot, h \otimes e)h' \otimes e'$ is zero and, hence, that $R_X R_Y = 0$ for all $X, Y$.

**Remark 1.** If $S \neq 0$ then the Jordan normal form of the Jacobi operators $R_X$ depends on the direction $X$. In fact $R_X = 0$ if $X \in H \otimes E^+$ and, if $S \neq 0$, then there exists $X \in H \otimes E^-$ such that $R_X \neq 0$.

**Remark 2.** In dimension 4, pseudo-Riemannian manifolds satisfying the Osserman condition pointwise are characterized as self-dual Einstein 4-manifolds [1]. This implies that symmetric Ricci-flat Osserman spaces of dimension 4 are the same as symmetric ($\varepsilon$)-hyper-Kähler manifolds. Similarly, *complex* symmetric Ricci-flat Osserman spaces of dimension 4 are the same as *complex* symmetric hyper-Kähler manifolds. It follows from Theorem 1 that any complex hyper-Kähler symmetric space of dimension 4 is defined by a quartic polynomial $S = e^4$, $e \in E$. Up to isomorphism, there are only two such manifolds: the flat one corresponding to $e = 0$ and the non-flat one corresponding to $e \neq 0$. In the real case, it follows from Theorem 2 that any 4-dimensional hyper-Kähler symmetric space is flat. However, by Theorem 3 there exists two non-flat para-hyper-Kähler symmetric spaces which correspond to the polynomials $\pm e^4$, $e \in E_0$. These manifolds occur in [3] as examples of Osserman spaces of signature $(2, 2)$.

## References


[1] D.V. Alekseevsky, N. Blažić, N. Bokan, Z. Rakić, Self-duality and pointwise Osserman manifolds, Arch. Math. (Brno) 35 (3) (1999) 193–201.

[2] D.V. Alekseevsky, V. Cortés, Classification of indefinite hyper-Kähler symmetric spaces, Asian J. Math. 5 (4) (2001) 663–684, math.DG/0007189.

[3] N. Blažić, N. Bokan, Z. Rakić, Osserman pseudo-Riemannian manifolds of signature (2, 2), J. Aust. Math. Soc. 71 (3) (2001) 367–395.

[4] V. Cortés, C. Mayer, T. Mohaupt, F. Saueressig, Special geometry of Euclidean supersymmetry I: vector multiplets, J. High Energy Phys. JHEP03 (2004), 028, hep-th/0312001.